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The fundamental optimal relation and the bounds of power output and efficiency for an irreversible Carnot engine

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Abstract. By using the finite-time thermodynamic method, the optimal performances of a Carnot engine under the influence of thermal resistance, heat loss and other irreversibilities are studied and the fundamental optimal relation for the engine is derived. Thus, a new theory for irreversible Carnot cycles is established, together with some significant discussions. The results obtained here are more general and useful than those in the relevant literature.

1. Introduction

According to classical thermodynamics, the upper bound on the efficiency of a heat engine is the so-called Carnot efficiency

$$\eta_C = 1 - T_L/T_H \tag{1}$$

where T_H and T_L are, respectively, the temperatures of the hot and cold reservoirs between which the heat engine operates. In fact, η_C is invariably greater than the efficiency of real heat engines and hence is of very limited practical value, because it corresponds to a reversible operation, i.e. it is an infinitely slow operation and thus has zero power output. No practical engineer wants to design or build an engine which runs infinitely slowly without producing power. Therefore, it is necessary to determine a new upper bound on the efficiency of a heat engine by using a new model. This has resulted in the advent of a new field: 'finite-time thermodynamics'.

Since the idea of finite-time thermodynamics was first advanced [1–5], considerable attention has been devoted to the problem of the best mode of operation of heat engines working in finite time. The most studied model consists of an endoreversible Carnot cycle with finite heat transfer in the isothermal branches [1, 6–9]. An upper bound can also be placed on the efficiency of a heat engine operating at its maximum power point: the so-called CA efficiency

$$\eta_{CA} = 1 - \sqrt{T_L/T_H} \tag{2}$$

where the sole source of irreversibility in the cycle is linear finite-rate heat transfer between the working fluid and its two heat reservoirs. η_{CA} has a more realistic significance than that of the reversible Carnot engine provided by classical thermodynamics because of its maximum power output.

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However, there are other sources of irreversibility in a real heat engine besides the irreversibility of finite-rate heat transfer. A complete treatment must take all loss mechanisms into account.

Recently, some new models of irreversible Carnot engines which include other irreversibilities besides thermal resistance have been established [10–14] and some significant conclusions about irreversible Carnot engines have been deduced. However, these models should be refined and developed further.

In this paper, we study the optimal performances of Carnot engines under the influence of thermal resistance, heat loss and other irreversibilities using a new irreversible Carnot cycle model. The fundamental optimal relation of the engine is derived, and the different characteristics of the influences of various irreversibilities are expounded. Some new significant conclusions which are more general and useful than those in the relevant literature are obtained. These form a new theory of the irreversible Carnot cycle, from which all the conclusions concerning a reversible Carnot engine, an endoreversible Carnot engine only affected by thermal resistance and an irreversible Carnot engine affected by various irreversibilities can be deduced. Thus it can serve as a guide to the evaluation of existing engines and the optimal design of future engines.

2. A new irreversible Carnot engine model

Consider the class of irreversible Carnot engines shown in figure 1, which satisfy the following four conditions:

(i) The cycle of the engine consists of two isothermal and two adiabatic processes. The temperatures of the working fluid in the hot and cold isothermal processes are, respectively, T_1 and T_2, and the times of the two isothermal processes are, respectively, t_1 and t_2. The temperatures of the hot and cold heat reservoirs are, respectively, T_H and T_L.

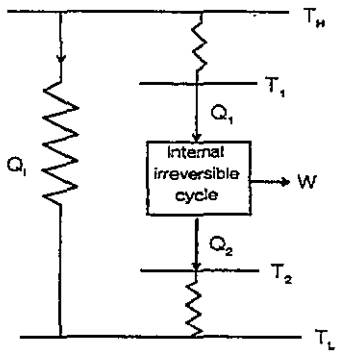

Figure 1. Schematic diagram of an irreversible Carnot heat engine.

(ii) There is thermal resistance between the working fluid and two heat reservoirs, and the two thermal conductances are, respectively, α and β. All heat transfer is assumed to be linear in temperature differences, that is, Newtonian. Therefore, we can write the heat Q_1 absorbed from the hot reservoir and the heat Q_2 released to the cold reservoir by the working fluid per cycle as follows

$$Q_1 = \alpha(T_H - T_1)t_1 \tag{3}$$

$$Q_2 = \beta(T_2 - T_L)t_2. \tag{4}$$

(iii) The heat Q_i lost from the hot reservoir to the cold reservoir per cycle is given by

$$Q_i = C_i(T_\mathrm{H} - T_\mathrm{L})\tau \tag{5}$$

where C_i is the coefficient of the heat loss, and τ is the cyclic time.

(iv) Besides thermal resistance and heat loss, there are other irreversibilities in the cycle. Because the total effect of the other irreversibilities can be characterized by increasing the change in entropy in the cold reservoir under a given absorbed heat Q_1, we can take $\Delta S_2'$ and ΔS_2 as the changes in entropy in the cold reservoir during a cycle only affected by thermal resistance and also by the other irreversibilities except heat loss, respectively. Thus, we can introduce a parameter

$$I_0 \equiv \Delta S_2/\Delta S_2' = Q_2/Q_2' \geqslant 1 \tag{6}$$

to characterize the other irreversibilities, where Q_2' is the heat released to the cold reservoir in a cycle only affected by thermal resistance. It is evident that there are no other irreversibilities except the thermal resistance when $I_0 = 1$.

Using such a model, the fundamental optimal relation, i.e. the optimal relation between the power output and efficiency, for an irreversible Carnot heat engine can be derived, and from this relation, the optimal performance of the engine can be discussed.

3. *The fundamental optimal relation*

According to the engine model described above, the total heat Q_H released from the hot reservoir and the total heat Q_L transferred to the cold reservoir are, respectively,

$$Q_\mathrm{H} = Q_1 + Q_i \tag{7}$$

and

$$Q_\mathrm{L} = Q_2 + Q_i. \tag{8}$$

On the other hand, according to the second law of thermodynamics, for an endoreversible Carnot cycle, one has

$$T_2/T_1 = Q_2'/Q_1. \tag{9}$$

Thus, from equation (6), we can get

$$T_2/T_1 = Q_2/(I_0Q_1). \tag{10}$$

By using equations (10) and (7), the efficiency of the engine without heat loss may be expressed as

$$\eta_0 = (Q_1 - Q_2)/Q_1 = 1 - I_0T_2/T_1 \tag{11}$$

and the efficiency of the engine with heat loss may be expressed as

$$\eta = \frac{Q_1 - Q_2}{Q_\mathrm{H}} = \left(1 - \frac{Q_2}{Q_1}\right)\frac{Q_1}{Q_\mathrm{H}} = \frac{Q_1\eta_0}{Q_1 + Q_i} = \frac{\eta_0 Q_1/\tau}{Q_1/\tau + Q_i/\tau}. \tag{12}$$

From equation (12), we can obtain the power output

$$P = (Q_1/\tau)\eta_0 = [(Q_1 + Q_i)/\tau]\eta. \tag{13}$$

Using equations (3), (4), (6) and (11), one obtains

$$Q_1/\tau = \frac{Q_1}{\gamma(t_1 + t_2)} = \frac{\alpha}{\gamma}\left[\frac{1}{T_\mathrm{H} - I_0T_2/(1 - \eta_0)} + \frac{\alpha(1 - \eta_0)}{\beta(T_2 - T_\mathrm{L})}\right]^{-1} \tag{14}$$

where $\gamma = \tau/(t_1 + t_2)$. When the two adiabatic processes are assumed to proceed in negligible time [6–9], $\gamma = 1$.

From equation (14) and using the extremal condition $\partial\eta_0/\partial T_2 = 0$, we can find that when

$$T_2 = \frac{T_L[1 + \sqrt{I_0\alpha/\beta}T_H/(T_H - Q_1/(K\tau))]}{1 + \sqrt{I_0\alpha/\beta}} \tag{15}$$

η_0 attains a maximum at a given Q_1/τ because $(\partial^2\eta_0/\partial T_2^2) < 0$ in such a case. Substituting equation (15) into equation (14), one can obtain the optimal relation between η_0 and Q_1/τ:

$$\eta_0 = 1 - I_0T_L/[T_H - Q_1/(K\tau)] \tag{16}$$

where $K = \alpha/[\gamma(1 + \sqrt{I_0\alpha/\beta})^2]$. Again, substituting equation (16) into equation (12), we obtain the optimal relation between η and Q_1/τ:

$$\eta = \frac{Q_1/\tau}{Q_1/\tau + Q_i/\tau}\left[1 - \frac{I_0T_L}{T_H - Q_1/(K\tau)}\right]. \tag{17}$$

Eliminating Q_1/τ from equations (17) and (13) and, using equation (5), one can obtain the fundamental optimal relation of an irreversible Carnot engine:

$$\begin{aligned}(1-\eta)P^2 - [K(T_H - I_0T_L) + C_i(2-\eta)(T_H - T_L) - KT_H\eta]\eta P \\ + C_i(T_H - T_L)[K(T_H - I_0T_L) + C_i(T_H - T_L)]\eta^2 = 0.\end{aligned} \tag{18}$$

4. Discussion and conclusion

(i) The characteristic curve of power P against efficiency η for a Carnot engine including thermal resistance, heat loss and other irreversibilities described by equation (18) is a loop line passing through the zero point shown as curve I in figure 2. On such a curve, there is a maximum power point (point A in figure 2) at which the power output is at a maximum P_{max} and the corresponding efficiency is η_m, and a maximum efficiency point (point B in figure 2) at which the efficiency is at a maximum η_{max} and the corresponding power output is P_m. η_{max} is different from the Carnot efficiency η_C and, in general, η_{max} is far less than η_C, but is in close proximity to η_m. These results mirror the observed performance of a real heat engine quite well [14].

(ii) From curve I in figure 2, it can be seen that the efficiency η and power output P of the engine are less than those at the maximum power point A when $\eta < \eta_m$; and η and P are less than those at the maximum efficiency point B when $P < P_m$. Therefore, the working states of $\eta < \eta_m$ and $P < P_m$ are not the optimal operating states of a real heat engine and the rational regions of the operating state should be set between the maximum power point and the maximum efficiency point, i.e. the negative slope regions of the characteristic curve P against η, shown as the arc AB in figure 2. Thus, two new criteria for finite-time thermodynamics can be established for the selection of an optimal operating parameter for a real heat engine, that is the efficiency η and power output P of the engine should satisfy the following two equations:

$$\eta_m \leqslant \eta \leqslant \eta_{max} \tag{19}$$

$$P_m \leqslant P \leqslant P_{max} \tag{20}$$

respectively. This shows that the four parameters η_m, η_{max}, P_m and P_{max} which determine the lower and upper bounds of the efficiency and power output are four important performance parameters of an irreversible Carnot engine. They are important guides and reference for the optimal design and the selection of optimal operating states of a real heat engine. Using

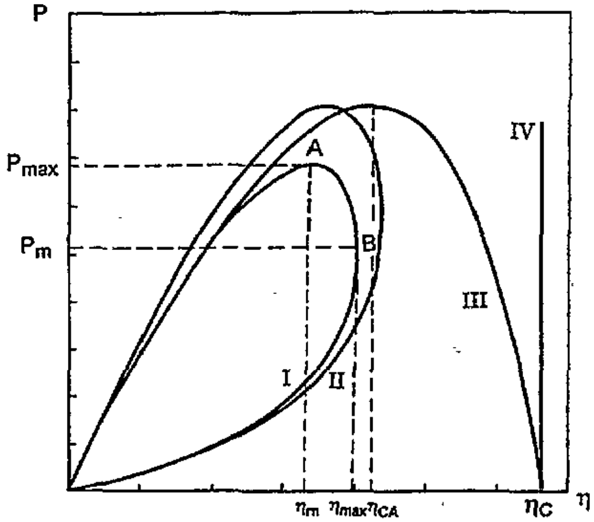

Figure 2. The power output against efficiency curves of an irreversible Carnot heat engine: curve I, for a Carnot engine including thermal resistance, heat loss and other irreversibilities; curve II, for a Carnot engine including thermal resistance and heat loss; curve III, for a Carnot engine only affected by thermal resistance; and curve IV, for a reversible Carnot engine.

equation (18) and the extremal conditions, the following four parameters can be derived (a detailed derivation is given in the appendix):

$$\eta_m = \left(1 - \sqrt{I_0 T_L/T_H}\right)^2 \Big/ \left(1 + \rho\eta_C - \sqrt{I_0 T_L/T_H}\right) \tag{21}$$

$$\eta_{max} = \rho\eta_C \left[\sqrt{1 + \frac{1 - I_0 T_L/T_H}{\rho\eta_C}} - \sqrt{I_0 T_L/T_H}\right]^2 /(1 + \rho\eta_C)^2 \tag{22}$$

$$P_m = K T_H \frac{\rho^2\eta_C^2\sqrt{1 + (1 - I_0 T_L/T_H)/\rho\eta_C}[\sqrt{1 + (1 - I_0 T_L/T_H)/\rho\eta_C} - \sqrt{I_0 T_L/T_H}]^2}{(1 + \rho\eta_C)[\rho\eta_C\sqrt{1 + (1 - I_0 T_L/T_H)/\rho\eta_C} + \sqrt{I_0 T_L/T_H}]} \tag{23}$$

$$P_{max} = K\left(\sqrt{T_H} - \sqrt{I_0 T_L}\right)^2 \tag{24}$$

where $\rho = C_i/K$.

(iii) When $I_0 = 1$, equation (18) becomes

$$(1-\eta)P^2 - \{(T_H - T_L)[K_1 + C_i(2-\eta)] - K_1\eta T_H\}\eta P + C_i(K_1 + C_i)(T_H - T_L)^2\eta^2 = 0 \tag{25}$$

where K_1 is the K for $I_0 = 1$. Equation (25) is just the optimal relation of the efficiency and power output of an irreversible Carnot engine only affected by thermal resistance and heat loss [11]. *Its characteristic curve P against η is also loop shaped, as shown by curve* II in figure 2. However, one can see that equation (18) is more useful and general than equation (25), because the former considers not only thermal resistance and heat loss but also the other irreversibilities of the heat engine, so that the characteristics of power output and efficiency of a real heat engine can be reflected more accurately. As long as an accurate estimate of I_0 can be obtained, some more satisfactory results can be obtained.

If $I_0 = 1$ and $C_i = 0$, equation (18) can be simplified as

$$P = K_1\eta[T_H - T_L/(1-\eta)]. \tag{26}$$

Equation (26) is just the fundamental optimal relation of an endoreversible Carnot engine [8, 9] and its characteristic curve P against η is shown as curve III in figure 2. It is not a loop line, and the maximum efficiency is the Carnot efficiency η_C which corresponds to a zero power output. Thus, there is no criterion for finite-time thermodynamics described by equation (20) for an endoreversible Carnot engine.

Besides $I_0 = 1$ and $C_i = 0$, if $\alpha \to \infty$ and $\beta \to \infty$ (thus $K_1 \to \infty$), i.e. the thermal resistance can also be neglected, the engine becomes reversible. In such a case, equation (18) can be simplified to

$$\eta = 1 - T_L/T_H = \eta_C \tag{27}$$

and the characteristic curve P against η is a straight line, running parallel with the vertical axis and cutting the horizontal axis at η_C. This shows that the theory of irreversible Carnot cycle established in this paper is also suitable for reversible Carnot cycles.

(iv) The characteristic curves P against η of a Carnot cycle under the four different cases shown in figure 1 clearly reflect the fact that different sources of irreversibilities result in different influences on the performance of the cycle.

First, for a reversible Carnot cycle, the P against η curve is a straight line located at $\eta = \eta_C$. The efficiency of the cycle is only dependent on the temperatures of the hot and cold reservoirs and it is not relevant to the power output. This indicates that the power output of a reversible Carnot cycle may be an arbitrary value, which is determined by the cyclic time τ for a given work output. If $\tau \to \infty$, then $P = 0$, i.e. the point $(0, \eta_c)$ on the horizontal axis in figure 2. As τ decreases, P increases, but η is invariant such that the P against η curve is a straight line from $P = 0$ to $P \to \infty$, which corresponds to τ from ∞ to 0.

Second, for an endoreversible Carnot cycle, thermal resistance exists, i.e. α and β are finite. In such a case, if the cycle attains Carnot efficiency η_C, the cyclic time τ would be infinite because the temperature differences between the working fluid and the two heat reservoirs need to be infinitesimal, so that the power output P would be zero. In other words, $\eta = \eta_C$ appears only at the zero power point on the P against η curve when the cycle has thermal resistance. For the cycle with a power output, the period τ must be finite and thus the temperature differences between the working fluid and the two heat reservoirs must also be finite in order to transfer an amount of heat. Consequently, the efficiency η of the cycle is less than η_C, and the shorter the period τ is, the larger the temperature differences $T_H - T_1$ and $T_2 - T_L$ for heat transfer are and the further away η is in value from η_c. When $T_1 = T_2$, the efficiency and power output both reach zero. Thus, besides the original zero power point $\eta = \eta_C$, there is a new zero power point at which the efficiency equals zero as well. At the same time, a maximum power point appears on the P against η curve (see curve III in figure 2). These results indicate clearly that the effect of thermal resistance on a Carnot cycle changes the P against η curve from a straight line (for a reversible cycle, curve IV in figure 2) into a parabola (curve III in figure 2), and the shorter the period is, the larger the effect is.

Curzon and Ahlborn [1] studied only the maximum power point of the endoreversible Carnot cycle. They derived the maximum power output P_{max} and the corresponding efficiency η_m, i.e. η_{CA}, of the cycle, which makes a contribution to the establishment and development of a new subject—finite-time thermodynamics.

Third, when the cycle has heat loss, i.e. $C_i \neq 0$, besides thermal resistance, one can see that the quantity of heat loss is in direct proportion to the period τ from equation (5). Therefore, for a given heat absorbed by the cycle, the longer the period is and, hence, the larger the loss of heat is. When the period approaches infinity, not only does the power output but also the efficiency approaches zero, such that the end point of the P against η curve for long period also tends to the zero point. This clearly explains why the P against η curve of an endoreversible Carnot cycle changes from a parabola (curve III in figure 2) into a loop (curve II in figure 2) under the influence of heat loss. For an endoreversible Carnot cycle with heat loss, the maximum efficiency is not the Carnot efficiency η_C and, generally, it is far less than η_C. This is just an important observed characteristic of real heat engines. The above results show that the effect of heat loss on the performance of a long-period cycle is more serious. This is the very opposite of what the thermal resistance effected.

Fourth, if there are any other irreversibilities in the cycle besides thermal resistance and heat loss, the P against η curve is still a loop line passing through the zero point (curve I in figure 2). However, the maximum power output P_{max} and the corresponding efficiency η_m, and the maximum efficiency η_{max} and the corresponding power output P_m of the cycle are all less than those of a cycle which does not have the other irreversibilities. In other words, the power output P and efficiency η of the cycle were correspondingly decreased by the effect of the other irreversibilities.

To sum up, all sources of the irreversibilities in the cycle produce an effect on the power output and efficiency, and they each have different characteristics. Equation (18) gives a comprehensive description and figure 2 gives a vivid illustration. This has contributed to a better understanding of the effect of the various irreversibilities on the performance of real heat engines.

(v) Some authors have introduced a parameter [15, 16]

$$I_S = \Delta S_{out}/\Delta S_{in} \tag{28}$$

to describe the degree of internal irreversibility resulting from the working fluid, where ΔS_{in} and ΔS_{out} are, respectively, the changes in entropy of the working fluid in two isothermal branches at temperatures T_1 and T_2, and they are defined as positive. Obviously, for an endoreversible Carnot cycle, $\Delta S_{in} = \Delta S_{out}$ and then $I_S = 1$. But, for an irreversible Carnot cycle including the internal irreversibilities, so long as the two adiabatic processes of the cycle are reversible, ΔS_{in} still equals ΔS_{out} and $I_S = 1$. It is thus clear that the degree of internal irreversibility I_S can describe only the internal irreversibilities of the two adiabatic processes, but not the two isothermal processes for an irreversible Carnot cycle. Therefore, when $I_S = 1$, the cycle may not be endoreversible, and the cyclic model introduced in some references to include the irreversibilities of the finite-rate heat transfer between the heat engine and its reservoirs, heat loss between the reservoirs and using I_S to describe the internal dissipations of the working fluid, is not a general cyclic model of the irreversible Carnot heat engine. It cannot describe all internal irreversibilities of the cycle, thus it is rather limited.

(vi) Because real heat engines include thermal resistance, heat loss and other irreversibilities, equation (18) in which all the irreversibilities in the engine are included mirrors fairly the observed performances of real engines. From equation (18), the various optimal relations and the bounds of performances can be derived, so that the optimal performances of the engine can be discussed. For example, using the general relation among the rate of energy loss ΔP, power P and efficiency η [17]

$$\eta = \eta_C/(1 + \Delta P/P) \tag{29}$$

one can find the minimum rate of energy loss $(\Delta P)_{min}$ for a Carnot engine at a given power output P from equation (18). This is the minimum irreversible loss which cannot be avoided for an irreversible Carnot engine at a given power output. The main difference between finite-time thermodynamics and classical thermodynamics is that the minimum irreversible loss which cannot be avoided in a cycle can be found in the former but not in the latter. Therefore, finite-time thermodynamics is more significant for real heat engines and the development of the theory of an irreversible Carnot cycle is an important step in the development of finite-time thermodynamics.

Acknowledgment

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Appendix

If we take equation (18) as a quadratic equation of the power output P

$$AP^2 + BP + C = 0 \tag{A1}$$

then we have

$$A = 1 - \eta \tag{A2}$$

$$B = -[K(T_H - I_0 T_L) + C_i(2-\eta)(T_H - T_L) - KT_H\eta]\eta \tag{A3}$$

$$C = C_i(T_H - T_L)[K(T_H - I_0 T_L) + C_i(T_H - T_L)]\eta^2 \tag{A4}$$

and that the maximum efficiency of the heat engine satisfies the following equation (extremal condition)

$$B^2 - 4AC = 0 \tag{A5}$$

and the corresponding power output P_m is given by

$$P_m = -B/(2A). \tag{A6}$$

Equation (A5) is a quadratic equation of η, from which we obtain the maximum efficiency η_{max} as expressed by equation (22). Substituting equation (22) into equation (A6), we obtain P_m as expressed by equation (23).

On the other hand, if we take equation (18) as a quadratic equation of the efficiency η

$$A'\eta^2 + B'\eta + C' = 0 \tag{A7}$$

then we have

$$A' = C_i(T_H - T_L)[K(T_H - I_0 T_L) + C_i(T_H - T_L)] + [C_i(T_H - T_L) + KT_H]P \tag{A8}$$

$$B' = -[K(T_H - I_0 T_L) + 2C_i(T_H - T_L)]P - P^2 \tag{A9}$$

$$C' = P^2 \tag{A10}$$

and that the maximum power output of the heat engine satisfies the following equation (extremal condition)

$$B'^2 - 4A'C' = 0 \tag{A11}$$

and the correspondent efficiency η_m is given by

$$\eta_m = -B'/(2A'). \tag{A12}$$

Equation (A11) is a quadratic equation of P, from which we obtain the maximum power output P_{max} as expressed by equation (24). Substituting equation (24) into equation (A12), we obtain η_m as expressed by equation (21).

References

[1] Curzon F L and Ahlborn B 1975 *Am. J. Phys.* **43** 22
[2] Andresen B, Salamon P and Berry R S 1984 *Phys. Today* **37** (9) 62
[3] Andresen B, Berry R S, Ondrechen M J and Salamon P 1984 *Acc. Chem. Res.* **17** 266
[4] Sieniutyze S and Salamon P 1991 Finite time thermodynamics *Advances in Thermodynamics* vol 4 (London: Taylor and Francis)
[5] Chen L and Yan Z 1987 *Nature J.* **10** 825 (in Chinese)
[6] Rubin M H 1979 *Phys. Rev.* A **19** 1272
[7] Salamon P and Nitzan A 1981 *J. Chem. Phys.* **74** 3546
[8] Yan Z 1985 *J. Engineering Thermophysics* **6** 1 (in Chinese)
[9] Chen L and Yan Z 1989 *J. Chem. Phys.* **90** 3740
[10] Bejan A 1988 *Int. J. Heat Mass Transfer* **31** 1211
[11] Chen L, Sun F and Chen W 1993 *Kexue Tongbao* **38** 480 (*Bull. Sci.* in Chinese)
[12] Yan Z 1991 *J. Xiamen Univ.* **30** 25 (in Chinese)
[13] Andresen B, Salamon P and Berry R S 1977 *J. Chem. Phys.* **66** 1571
[14] Gordon J M and Huleihil M 1991 *J. Appl. Phys.* **69** 1; 1992 *J. Appl. Phys.* **72** 829
[15] Wu C and Kiang R L 1992 *Energy* **17** 1173
[16] Ibrahim O M, Klein S A and Mitchell J W 1991 *Trans. ASME J. Engng. Gas Trub. Pow.* **113** 514
[17] Liao X and Yan Z 1994 *J. Xiamen Univ.* **33** 771 (in Chinese)